

Unitarity Bounds and the Cuspy Halo Problem

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Conventional Cold Dark Matter cosmological models predict small scale structures, such as cuspy halos, which are in apparent conflict with observations. Several alternative scenarios based on modifying fundamental properties of the dark matter have been proposed. We show that general principles of quantum mechanics, in particular unitarity, imply interesting constraints on two proposals: collisional dark matter proposed by Spergel & Steinhardt, and strongly annihilating dark matter proposed by Kaplinghat, Knox & Turner. Efficient scattering required in both implies $m \lesssim 12$ GeV and $m \lesssim 25$ GeV respectively. The same arguments show that the strong annihilation in the second scenario implies the presence of significant elastic scattering, particularly for large enough masses. Recently, a variant of the collisional scenario has been advocated to satisfy simultaneously constraints from dwarf galaxies to clusters, with a cross section that scales inversely with velocity. We show that this scenario likely involves super-elastic processes, and the associated kinetic energy change must be taken into account when making predictions. Exceptions and implications for experimental searches are discussed.

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I. INTRODUCTION

There is a long history of efforts to constrain dark matter properties from galactic structure (e.g. [1]). Recent numerical simulations [2] sharpen the predictions of Cold Dark Matter (CDM) structure formation models, and apparent discrepancies with the observed properties of structures from galactic to cluster scales are uncovered. The main one that has attracted a lot of attention is the cuspy halo problem, namely that CDM models predict halos that have a high density core or have an inner profile that is too steep compared to observations ([3], but see also [4]). This has encouraged several proposals that dark matter might have properties different from those of conventional CDM (see [5] and summary therein).

On the other hand, general principles of quantum mechanics impose non-trivial constraints on some of these models. We focus here on the proposals of collisional or strongly self-interacting dark matter (SIDM) by [6] and of strongly annihilating dark matter (SADM) by [7]. Both require high level of interaction by particle physics standard: an elastic scattering cross-section of $\sigma_{\text{el.}} \sim 10^{-24}(m_X/\text{GeV})\text{cm}^2$ for the former and an annihilation cross-section of $\sigma_{\text{ann.}} v_{\text{rel}} \sim 10^{-28}(m_X/\text{GeV})\text{cm}^2$ for the latter, where m_X is the particle mass, and v_{rel} is the relative velocity of approach. The proposed dark matter is therefore quite different from usual candidates such as the axion or neutralino. We show that the unitarity of the scattering matrix, together with a few reasonable assumptions, imposes interesting particle mass bounds as well as other physical constraints. This is done while making minimal assumptions about the nature of the interactions. Our results complement constraints from experiments or astrophysical considerations e.g. [8].

Griest and Kamionkowski [9] previously derived similar mass bounds related to the freeze-out density of thermal relics, assuming 2-body final states. In §II, we provide a general derivation for arbitrary final states using the classic optical theorem [10]. We summarize our findings in §III, discuss exceptions to our bounds, and other solutions to the cuspy halo problem.

II. DERIVING THE UNITARITY BOUNDS

Different versions of the unitarity bounds can be found in many textbooks, and can be most easily understood using non-relativistic quantum mechanics (e.g. [11]), which is probably adequate for our purpose. However, the results and derivation given here might be of wider interest e.g. for estimating thermal relic density. Here we follow closely the field theory treatment of [12].

The optical theorem [10] is a powerful consequence of the unitarity of the scattering matrix S , i.e. $S^\dagger S = 1$, which implies $(1 - S)^\dagger(1 - S) = (1 - S^\dagger) + (1 - S)$, or

$$\int d\gamma \langle \beta | 1 - S | \gamma \rangle \langle \gamma | 1 - S^\dagger | \alpha \rangle = 2 \text{Re} \langle \beta | 1 - S | \alpha \rangle \quad (1)$$

where α and β represent two specified states and γ represents a complete set of states with measure $d\gamma$. Using the definition of the scattering amplitude $A_{\beta\alpha}$

$$\langle \beta | 1 - S | \alpha \rangle \equiv -i(2\pi)^4 \delta^4(p_\beta - p_\alpha) A_{\beta\alpha} \quad (2)$$

where p_β and p_α are the total four-momenta, one obtains

$$\int d\gamma (2\pi)^4 \delta^4(p_\alpha - p_\gamma) |A_{\gamma\alpha}|^2 = 2 \text{Im} A_{\alpha\alpha} \quad (3)$$

if $\beta = \alpha$ in eq. (1). We are interested in the case where α represents a 2-body state of $X + X$ or $X + \bar{X}$ approaching each other. The final state γ , on the other hand, is completely general, and the integration over γ covers the entire spectrum of possible final states. To be more precise, suppose $|\alpha\rangle = |k_1, s_1^z; k_2, s_2^z; n\rangle$ where s_1^z and s_2^z represent the spin-states of the two incoming particles with spins s_1 and s_2 (in our particular case, $s_1 = s_2$) while k_1 and k_2 are their respective 4-momenta, and n labels the particle types (e.g. mass, etc.). Recalling that $d\sigma/d\gamma \propto |A_{\gamma\alpha}|^2$, eq. (3) gives in the center of mass frame (adopted hereafter i.e. $\mathbf{k}_1 + \mathbf{k}_2 = 0$):

$$\int d\gamma \frac{d\sigma}{d\gamma}(\alpha \rightarrow \gamma) = \frac{\text{Im } A_{\alpha\alpha}}{2(E_1 + E_2)|\mathbf{k}_1|} \quad (4)$$

where the left hand side is exactly the total cross-section. This is the optical theorem. It states that the total cross-section for scattering from a two-body initial state *to all possible* final states equals the imaginary part of the two-body *to two-body* forward scattering amplitude.

To use this theorem, we expand the scattering amplitude in terms of partial waves i.e. states labeled as $|\mathbf{k}_{\text{tot.}}, E_{\text{tot.}}, j, j^z, \ell, s, n\rangle$ where $\mathbf{k}_{\text{tot.}}$ is the total linear momentum ($= 0$ in the center of mass frame), $E_{\text{tot.}}$ is the total energy, j is the total angular momentum, j^z is its z-component, ℓ is the orbital momentum and s is the total spin. Inserting appropriate complete sets of partial wave outer products into eq. (2), we obtain

$$\begin{aligned} A_{\beta\alpha} &= 4i(2\pi)^2 [E'_1 + E'_2/|\mathbf{k}'_1|]^{\frac{1}{2}} [E_1 + E_2/|\mathbf{k}_1|]^{\frac{1}{2}} \quad (5) \\ &\sum_{j, j^z} \sum_{\ell', s', \ell, s} \langle \ell' s' n' | 1 - S | \ell s n \rangle_{j, E_{\text{tot.}}} \\ &\sum_{\ell'^z, s'^z} \langle s'^z | s_1^z s_2^z \rangle_{s'_1, s'_2} \langle j j^z | \ell'^z s'^z \rangle_{\ell', s'} \langle \hat{\mathbf{k}}_1 | \ell' \ell'^z \rangle \\ &\sum_{\ell^z, s^z} \langle s s^z | s_1^z s_2^z \rangle_{s_1, s_2}^* \langle j j^z | \ell^z s^z \rangle_{\ell, s}^* \langle \hat{\mathbf{k}}_1 | \ell \ell^z \rangle^* \end{aligned}$$

where the crucial assumption is that S is rotationally invariant and so j and j^z are conserved, in addition to energy conserving. The notation $\langle \ell' s' n' | 1 - S | \ell s n \rangle_{j, E_{\text{tot.}}}$ emphasizes that S is diagonal in j, j^z and $E_{\text{tot.}}$ but the j^z dependence drops out because S commutes with $J_x \pm iJ_y$. The inner products $\langle s s^z | s_1^z s_2^z \rangle_{s_1, s_2}$ and $\langle j j^z | \ell^z s^z \rangle_{\ell, s}$ give the Clebsch-Gordon coefficients, and $\langle \hat{\mathbf{k}}_1 | \ell \ell^z \rangle = Y_{\ell\ell^z}(\hat{\mathbf{k}}_1)$ is the spherical harmonic function. We assume $\hat{\mathbf{k}}_1 = \hat{\mathbf{z}}$ in which case $Y_{\ell\ell^z}(\hat{\mathbf{k}}_1) = \delta_{\ell^z, 0} \sqrt{2\ell+1/(4\pi)}$. The index β denotes a 2-body final state $|k'_1, s'_1; k'_2, s'_2; n'\rangle$.

Setting $\beta = \alpha$, and averaging over the spin-states (i.e. $(2s_1 + 1)^{-1}(2s_2 + 1)^{-1} \sum_{s_1^z, s_2^z}$) on both sides of eq. (4), the optical theorem, we obtain [12]:

$$\begin{aligned} \sigma_{\text{tot.}} &= \frac{2\pi}{|\mathbf{k}_1|^2(2s_1 + 1)(2s_2 + 1)} \sum_j (2j + 1) \quad (6) \\ &\sum_{\ell, s} \text{Re} \langle \ell s n | 1 - S | \ell s n \rangle_{j, E_{\text{tot.}}} \end{aligned}$$

This gives the total spin-averaged cross-section for scattering from $X + X$ or $X + \bar{X}$ to all possible final states.

For $X + \bar{X}$ annihilation, we exclude from the above the contribution due to elastic scattering (where type and mass of particles do not change i.e. $X + \bar{X} \rightarrow X + \bar{X}$, implying $|\mathbf{k}'_1| = |\mathbf{k}_1|$) [9]. To do so, we need the following expression for 2-body to 2-body scattering cross-section:

$$\frac{d\sigma}{d\beta} d\beta = \frac{|A_{\beta\alpha}|^2}{4(E_1 + E_2)|\mathbf{k}_1|} (2\pi)^4 \delta^4(p_\beta - p_\alpha) d\beta, \quad (7)$$

We average over initial spin states and integrate over outgoing momenta, but focus on the elastic contribution (n' in $|\beta\rangle = |k'_1, s'_1; k'_2, s'_2; n'\rangle$ is set to n in $|\alpha\rangle$) [12]:

$$\begin{aligned} \sigma_{\text{el.}} &= \frac{\pi}{|\mathbf{k}_1|^2(2s_1 + 1)(2s_2 + 1)} \sum_j (2j + 1) \quad (8) \\ &\sum_{\ell, s, \ell', s'} |\langle \ell' s' n | 1 - S | \ell s n \rangle_{j, E_{\text{tot.}}}|^2 \end{aligned}$$

The above is the total cross-section for elastic scattering (note: the same expression also describes $X + X \rightarrow X + X$ elastic scattering) that has to be subtracted from $\sigma_{\text{tot.}}$ to yield the total inelastic scattering cross-section, which is relevant for annihilation into all possible final states:

$$\begin{aligned} \sigma_{\text{inel.}} &= \frac{\pi}{k_1^2(2s_1 + 1)(2s_2 + 1)} \sum_j (2j + 1) \quad (9) \\ &\sum_{\ell, s} [1 - |\langle \ell s n | S | \ell s n \rangle|^2 - \sum_{\ell' \neq \ell, s' \neq s} |\langle \ell' s' n | 1 - S | \ell s n \rangle|^2] \end{aligned}$$

From eq. (6) & (9), we can derive two bounds:

$$\sigma_{\text{tot.}} \leq 4\pi[|\mathbf{k}_1|^2(2s_1 + 1)(2s_2 + 1)]^{-1} \sum_j \sum_{\ell, s} 2j + 1 \quad (10)$$

$$\sigma_{\text{inel.}} \leq \pi[|\mathbf{k}_1|^2(2s_1 + 1)(2s_2 + 1)]^{-1} \sum_j \sum_{\ell, s} 2j + 1 \quad (11)$$

The first inequality uses $|\langle \ell s n | S | \ell s n \rangle|^2 \leq 1$, obtained from $\int d\gamma \langle \ell s n | S^\dagger | \gamma \rangle \langle \gamma | S | \ell s n \rangle \geq |\langle \ell s n | S | \ell s n \rangle|^2$ and $S^\dagger S = 1$. A similar bound can be derived for $\sigma_{\text{el.}}$ as well, which coincides exactly with that for $\sigma_{\text{tot.}}$.

We pause to note that the above bounds assume only unitarity and the conservation of total energy and linear and angular momentum. No assumptions are made about the nature of the particles, whether they are composite or point-like. Nor do we assume the number of particles in the final states. To obtain useful limits from the bounds, we take the low velocity limit. Assuming the scattering amplitude $A_{\beta\alpha}$ is an analytic function of \mathbf{k}_1 as $\mathbf{k}_1 \rightarrow 0$ (exceptions will be discussed in §III), and noting that $k^\ell \langle \hat{\mathbf{k}} | \ell \ell^z \rangle$ is a polynomial function of \mathbf{k} , we expect the ℓ partial wave contribution to $A_{\beta\alpha}$ (eq. 5) to scale as $|\mathbf{k}_1|^\ell$. This means in the low velocity limit, as is relevant for our purpose (typical velocity dispersion in halos range from 10 to 1000 km/s $\ll c$), the $\ell = 0$ or s-wave contribution dominates. Setting $\ell = 0$ in eq. (10), (11):

$$\sigma_{\text{tot.}} \leq 16\pi/(m_X v_{\text{rel}})^2, \quad \sigma_{\text{inel.}} v_{\text{rel}} \leq 4\pi/(m_X^2 v_{\text{rel}}) \quad (12)$$

where $k_1^2 = m_X^2 |\mathbf{v}_2 - \mathbf{v}_1|^2/4 = m_X^2 v_{\text{rel}}^2/4$ is used. The second inequality agrees with [9]. Hence,

$$\sigma_{\text{tot.}} \leq 1.76 \times 10^{-17} \text{ cm}^2 \left[\frac{\text{GeV}}{m_X} \right]^2 \left[\frac{10 \text{ km s}^{-1}}{v_{\text{rel.}}} \right]^2 \quad (13)$$

$$\sigma_{\text{inel.}} v_{\text{rel.}} \leq 1.5 \times 10^{-22} \text{ cm}^2 \left[\frac{\text{GeV}}{m_X} \right]^2 \left[\frac{10 \text{ km s}^{-1}}{v_{\text{rel.}}} \right] \quad (14)$$

Furthermore, if $\sigma_{\text{inel.}}$ is bounded from below, say $\sigma_{\text{inel.}} \geq \sigma_{\text{ann.}}$, one can derive a lower bound on $\sigma_{\text{el.}}$ using eq. (8) & (9), and setting $\ell = 0$. Defining $\langle X \rangle_J \equiv [(2s_1 + 1)(2s_2 + 1)]^{-1} \sum_{j,\ell,s} (2j+1)X$, using S at the moment to denote $\langle \ell s n | S | \ell s n \rangle$, and noting that $\langle 1 \rangle_J = 1$ for $\ell = 0$, it can be shown $(\pi/k_1^2)(1 - \langle |S|^2 \rangle_J) \geq (\pi/k_1^2)(1 - \langle |S|^2 \rangle_J) \geq \sigma_{\text{inel.}} \geq \sigma_{\text{ann.}}$, which implies $\langle |S| \rangle_J \leq \sqrt{1 - k_1^2 \sigma_{\text{ann.}}/\pi}$. Also, $\sigma_{\text{el.}} \geq (\pi/k_1^2) \langle |1 - S|^2 \rangle_J \geq (\pi/k_1^2) \langle (1 - |S|^2) \rangle_J \geq (\pi/k_1^2)(1 - \langle |S|^2 \rangle_J)^2$. Combining, we have

$$\sigma_{\text{el.}} \geq (\pi/k_1^2) [1 - \sqrt{1 - k_1^2 \sigma_{\text{ann.}}/\pi}]^2 \quad (15)$$

This tells us that the elastic scattering cross-section cannot be arbitrarily small given a non-vanishing inelastic cross-section, e.g. via annihilation.

The above 3 bounds are the main results of this section. Two more results will be useful for our later discussions. For two-body to two-body processes, recall that the ℓ, ℓ' contribution to $A_{\beta\alpha}$ scales as $|\mathbf{k}_1|^\ell |\mathbf{k}'_1|^{\ell'}$. Using $d\sigma/d\Omega = |A_{\beta\alpha}|^2 (|\mathbf{k}'_1|/|\mathbf{k}_1|) / [64\pi^2 (E_1 + E_2)^2]$ (obtained from eq. 7 by integrating over β except for solid angle Ω), it can be seen that for elastic scattering, where $|\mathbf{k}'_1| = |\mathbf{k}_1|$,

$$d\sigma/d\Omega \rightarrow \text{const.} [1 + O(v_{\text{rel.}})] \quad (16)$$

as $|\mathbf{k}_1| \rightarrow 0$. For inelastic scattering where the system gains kinetic energy by losing rest mass (e.g. de-excitation of a composite particle or annihilation), since $|\mathbf{k}'_1|$ approaches a non-zero value as $|\mathbf{k}_1| \rightarrow 0$, we have

$$d\sigma/d\Omega \rightarrow (\text{const.}/v_{\text{rel.}}) [1 + O(v_{\text{rel.}})] \quad (17)$$

instead in the low velocity limit. The opposite case where the particle gains mass is discussed in [12].

III. DISCUSSION

We can derive the following four constraints for strongly self-interacting dark matter (SIDM) [6] and strongly annihilating dark matter (SADM) [7].

1. The range $\sigma_{\text{el.}} \sim 10^{-24} - 10^{-23} \text{ cm}^2 (m_X/\text{GeV})$ is given by [5] for SIDM to yield the desired halo properties. Using the lower $\sigma_{\text{el.}}$, and $v_{\text{rel.}} \sim 1000 \text{ km/s}$ (appropriate for clusters), eq. (13) tells us $m_X \lesssim 12 \text{ GeV}$ for SIDM.

2. The annihilation cross-section from [7], $\sigma_{\text{ann.}} v_{\text{rel.}} \sim 10^{-28} \text{ cm}^2 (m_X/\text{GeV})$, together with eq. (14) and $v_{\text{rel.}} \sim$

1000 km/s, gives us a bound of $m_X \lesssim 25 \text{ GeV}$ for strongly annihilating dark matter.

3. For SADM, efficient annihilation (a form of inelastic scattering) inevitably implies some elastic scattering as well. From eq. (15), and using $v_{\text{rel.}} \sim 1000 \text{ km/s}$ as before, we have

$$\sigma_{\text{el.}} \geq 4 \times 10^{-22} \text{ cm}^2 [\text{GeV}/m_X]^2 [1 - \sqrt{1 - 7 \times 10^{-5} (m_X/\text{GeV})^3}]^2 \quad (18)$$

Two simple limiting cases: when m_X is close to the upper bound of 25 GeV, $\sigma_{\text{el.}} \gtrsim 4 \times 10^{-22} \text{ cm}^2$; when m_X is small, $\sigma_{\text{el.}} \gtrsim 5 \times 10^{-31} \text{ cm}^2 (m_X/\text{GeV})^4$. Hence, elastic scattering is inevitable in this scenario, but can be reduced by having a sufficiently small mass.

4. Recent simulations suggest that the simplest version of SIDM fails to match simultaneously the observed halo properties from dwarf galaxies to clusters [13,5] (see also [14]), which have $v_{\text{rel.}}$ ranging over 3 orders of magnitude. It was suggested that an *elastic* scattering cross-section of $\sigma \propto 1/v_{\text{rel.}}$ might solve the problem. But as shown in eq. (16), elastic scattering generally implies $\sigma \rightarrow \text{constant}$ in the small velocity limit. Hence, $\sigma \propto 1/v_{\text{rel.}}$ likely requires *inelastic* processes. As eq. (17) shows, processes in which the net kinetic energy increases ($|\mathbf{k}'_1| > |\mathbf{k}_1|$ in c.o.m. frame) can give such a velocity dependence. SADM provides an example. More generally, the net kinetic energy increase (super-elasticity) must be taken into account when considering the viability of a model with $\sigma \propto 1/v_{\text{rel.}}$ e.g. it may delay core collapse and make the core larger. Note, however, the general considerations in the last section does not forbid an elastic cross-section that increases as $v_{\text{rel.}}$ decreases e.g. the $O(v_{\text{rel.}})$ term in eq. (16) can have a negative coefficient. A $1/v_{\text{rel.}}$ power-law may approximate such a cross-section, but likely only for a limited range of $v_{\text{rel.}}$. An example is the neutron-neutron scattering cross-section, which approaches a constant for $|\mathbf{k}_1| \lesssim 10^{-2} \text{ GeV}$, and scales as $1/v_{\text{rel.}}$ only for $10^{-2} \lesssim |\mathbf{k}_1| \lesssim 5 \times 10^{-2} \text{ GeV}$ [17].

It is helpful to mention here possible exceptions to the above limits. Our bounds are obtained from eq. (13) & (14), which are the $\ell = 0$ (s-wave) versions of eq. (10) & (11). The argument for putting $\ell = 0$ in the small velocity limit assumes the analyticity of $A_{\beta\alpha}$ at $\mathbf{k}_1 = 0$. The latter breaks down if the interaction is long-ranged, e.g. Coulomb scattering. This is unlikely to be relevant, because there are strong constraints on dark matter with such long ranged interaction [15]. Our argument for the dominance of s-wave scattering can also be invalid if there is a resonance. However, given that the scattering cross section should vary smoothly over three orders of magnitude in velocities from dwarfs to clusters, a resonance seems unlikely. Finally, the most likely situation in which the bounds break down is if the particle has a large enough size, or the interaction has a large enough effective range, R , such that $|\mathbf{k}_1|R > 1$ (e.g. see [16]). In such

cases, higher partial waves in addition to s-wave generally contribute, and $\sigma_{\text{tot.}} \lesssim 64\pi R^2$ and our arguments turn into a limit on R [9]. The condition $|\mathbf{k}_1|R > 1$ gives the most stringent constraint on R for $v_{\text{rel.}} = 10$ km/s, as appropriate for dwarf galaxies: $R \gtrsim 10^{-9} \text{ cm}(\text{GeV}/m_X)$. One can compare this with R for neutron-neutron scattering $\sim 10^{-13} \text{ cm}$ [17].

It is intriguing that halo structure might be telling us the elementary properties, in particular the mass, of dark matter. It is interesting that several proposals to address the cuspy halo problem, such as Warm Dark Matter [18] and Fuzzy Dark Matter [19] make explicit assumptions about the mass of the particles – $m_X \sim 1$ keV and $m_X \sim 10^{-22}$ eV respectively. For SIDM and SADM, astrophysical considerations generally only put constraints on the cross-section per unit mass. We have shown here that unitarity arguments imply a rather modest mass for both scenarios as well. It is also worth pointing out that our arguments, with suitable modification to take into account bose enhancement and multiple incoming particles, can be extended to cover dark matter in the form of a bose condensate, as has been proposed as yet another solution to the cuspy halo problem [20]. They generally require small masses as well $\lesssim 10$ eV.

A few issues are worth further investigation. Wandelt et al. [8] recently argued a version of SIDM, where the dark matter interacts strongly also with baryons, is experimentally viable, but requires $m_X \gtrsim 10^5$ GeV, or $m_X \lesssim 0.5$ GeV. Our bound here is inconsistent with the large mass region (but see exceptions above); experimental constraints on the low mass region will be very interesting ($\sigma_{\text{el.}} \lesssim 10^{-25} \text{ cm}^2$). It would be useful to find a micro-physics realization of the collisional scenario [21] or its variant where σ scales appropriately with velocity to match observations. The impact of inelastic collisions on halo structures is worth exploring in more detail. It is also timely to reconsider possible astrophysical solutions to the cuspy halo problem, such as the use of mass loss mechanisms [22]. We hope to examine some of these issues in the future.

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